WEEK SEVEN: CALCULATIONS IN CHAPTER 8
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1. NOTATIONS AND SETUP

Recall our previous set-up:

- $F(z) = G(z)/H(z)$; for us G and H are polynomials, but they can, in fact, be analytic functions provided H satisfies some conditions;
- $V = \mathbb{V}(H) =$ the variety of H, in $\mathbb{C}^d$;
- amoeba$(H) = \{\Re(\log z) : H(z) = 0\}$ in $\mathbb{R}^d$, where

$$\Re \log(z_1, \ldots, z_d) = (\log |z_1|, \ldots, \log |z_d|);$$

- B, a connected component of $\mathbb{R}^d \setminus$ amoeba(H). By the chapter on amoeba's, such components are in 1-to-1 correspondence with Laurent Series Expansions of $1/H$ (and hence of F);
- We let $F(z) = \displaystyle\sum_{r=-I}^{\infty} z_r z^r$ be the Laurent expansion of F which converges in B.

Now, we pause for an example of an amoeba.

Example 1. *We calculate amoeba$(1 - x - y)$. If we set $1 - x - y = 0$ then $\Re \log(x, y) = (\log |x|, \log |1 - x|)$. We consider various cases.*

(a) *($x \in \mathbb{R}, x \geq 2$) Then $\log |x| = \log(x) > 0$ and $\log |1 - x| = \log(x - 1) \geq 0$, so our amoeba is in the first quadrant. If $x = 2$ then we get the x-intercept $(\log 2, 0)$. As x gets large, we converge to $x = y$.*

(b) *($x \in \mathbb{R}, 1 < x < 2$) Here, $|1 - x|$ takes on values between 0 and 1, so we are in the fourth quadrant. As $1 - x \to 0$ as $x \to 1$, the $y-$axis is an asymptote of the function.*

(c) *By symmetry, we have the same results as above if we exchange x and y.*

(d) *Finally, we check if $(0, 0)$ is in the amoeba (i.e., if the amoeba is the inside or the outside of the curves found above). It's in the amoeba iff $|x| = 1$ and $|1 - x| = 1$ has a solution. But this is asking if two circles with radius 1 and centers 1 unit apart intersect in the complex plane intersect – which they do (in fact, twice).*

(e) *The amoeba crosses the line $y = x$ when $\log |x| = \log |1 - x|$, i.e., when $x = 1/2$. This gives the point $(-\log 2, -\log 2)$ in the amoeba.*

Putting all this together gives the following picture:

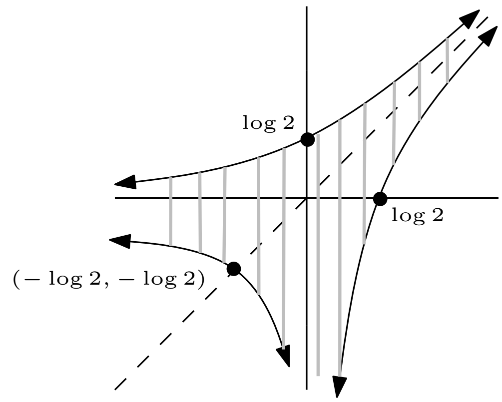

FIGURE 1. The amoeba of $1 - x - y$

What about the corresponding Laurent expansions? The power series solution at 0 converges for $|x + y| < 1$ (plus possibly on the boundary). If x, y are real and positive then both are at most 1, so they end up in the third quadrant after taking the $\Re \log$ map. Note also that the power series part will always contain the ray on $x = y$ starting from $(-r, -r)$ for some sufficiently large r as this corresponds to approaching the origin under the $\Re \log$ map.

We also had the following set-up:

- $r \in \mathbb{N}^d$ an index;
- The Cauchy Integrand $\omega = z^{-r-1}F(z)dz$, analytic in $\mathcal{M} = (\mathbb{C}^*)^d \setminus V$;
- $\hat{r} = r/|r|$, with $|r| = r_1 + \cdots + r_d$ being the l_1 norm;
- $T(x) = \exp(x + i \cdot \mathbb{R}^r)$.

2. WHAT DID SOPHIE DO?

Before we talk about asymptotics, we need to talk about exponential growth. Because of oscillations, given $\hat{r_*}$ we define

$$\overline{\beta}(\hat{r_*}) = \inf_{\mathcal{N}} \left(\limsup_{\substack{r \to \infty \\ \hat{r} \in \mathcal{N}}} \frac{\log |a_r|}{|r|} \right),$$

where $\mathcal{N}$ runs over a system of neighbourhoods whose intersection is $\{\hat{r}_*\}$.

Example 2. *Last time we looked at*

$$a_{rs} = \binom{r+s-1}{s} - \binom{r+s-1}{r}.$$

Then $F(x, y) = \sum a_{ij} x^i y^j = \frac{x-y}{1-x-y}$, the function whose amoeba we studied above. Letting $\hat{r}_* = (1/2, 1/2)$ we can calculate the Taylor expansion of $a_{r+\epsilon_1, r-\epsilon_2}$ in Maple to first order in ϵ_1, ϵ_2. Taking $r \to \infty$, Maple gives the limit as $\log 2$, so the exponential growth is 2.

Another thing we saw last time:

Definition 1. *Given $r \in \mathbb{R}^d$, we let $B^*(r) = \inf(-r \cdot x, x \in B)$.*

Which gave

Proposition 2. $\overline{\beta}(\hat{r_*}) \leq \beta^*(\hat{r}_*)$

Proof. By the definition of $\sum a_r z^r$, we have that if $z = \exp(x+iy)$ for $x \in B$ and any $y \in \mathbb{R}^d$, then the series converges. In particular, $|a_r z^r| \to 0$ for every $r \to \infty$. But

$$|z^r| = |z_1|^{r_1} \cdots |z_d|^{r_d} = e^{x \cdot r},$$

so for all $x \in B$ and $\epsilon > 0$ sufficiently small $|a_r| < \epsilon e^{-r \cdot x}$. This implies $\log |a_r| < -r \cdot x$ for all but finitely many r, so

$$\frac{\log |a_r|}{|r|} \leq \inf(-\hat{r} \cdot x),$$

for all but finitely many r. Taking the infimum over a system of neighbourhoods whose intersection is $\{\hat{r}_*\}$ gives the final result. ☐

Then Sophie proved some results about Morse Theory and defined critical points (we had a definition of critical(r) as the set of solutions to the critical point equations with respect to the direction $\hat{r}$).

3. Minimal Points

We refine the notion of critical points to get closer to those that actually contribute to the asymptotics.

3.1. **Minimal Points.** Critical points on the boundary $z \in \partial B$ are called minimal points, and the set of them is denoted minimal($\hat{r}_*$). By definition, minimal($\hat{r}_*$) $\subset$ critical($\hat{r}_*$) – the idea is that the minimal points are the only ones that can actually contribute to the asymptotics.

3.2. **Locally Oriented Points.** There is a further refinement of minimal points – these points are called locally oriented, and the set of them is denoted local($\hat{r}_*$). They are defined in Chapter 11, but we give an example here.

Example 3. *Let* $H = L_1 L_2 = (3 - x - 2y)(3 + 2x + y)$.

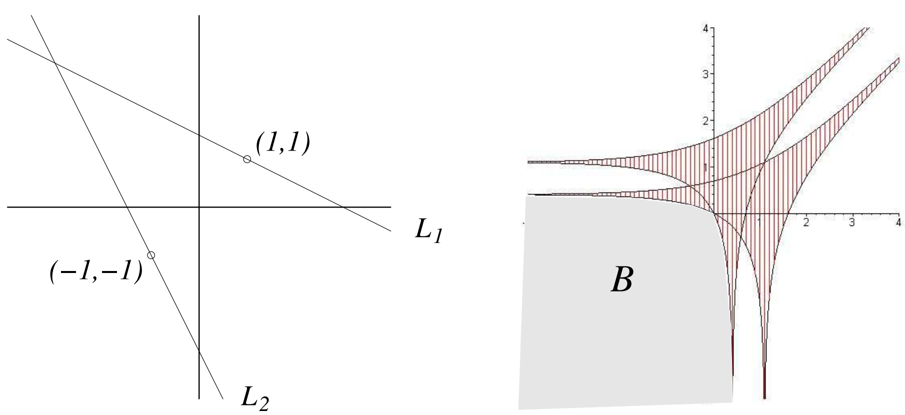

Figure 2. The varieties of L_1 and L_2 (restricted to the real plane) and the amoeba of H.

Now, $(1,1) \in L_1$ *and* $(-1,-1) \in L_2$*, and* $\Re \log(1,1) = (0,0) = \Re \log(-1,-1)$*. Thus,* $\Re \log V_{L_1}$ *and* $\Re \log V_{L_2}$ *intersect at* $(0,0)$*, but there is no corresponding intersection on the varieties themselves! Thus, although* $(0,0)$ *is a minimal point, we would not call it a locally oriented point.*

4. The gluing data

Finally, in Chapter 8.5 the text describes the quasi-local cycles by describing the topology near the critical points. Let $M = \mathbb{C}^d \setminus V$ and $S =$ a stratum of V. For $x \in S$, locally there is a product structure; i.e., for $\mathcal{N}$ a sufficiently small neighbourhood of x in B, $\mathcal{N}$ is diffeomorphic to $N \times B_k$, where k is the real dimension of B, B_k denotes the $k-$ball, and N is a normal slice $\mathcal{N} \cap P$, where P is the plane normal to S at x.

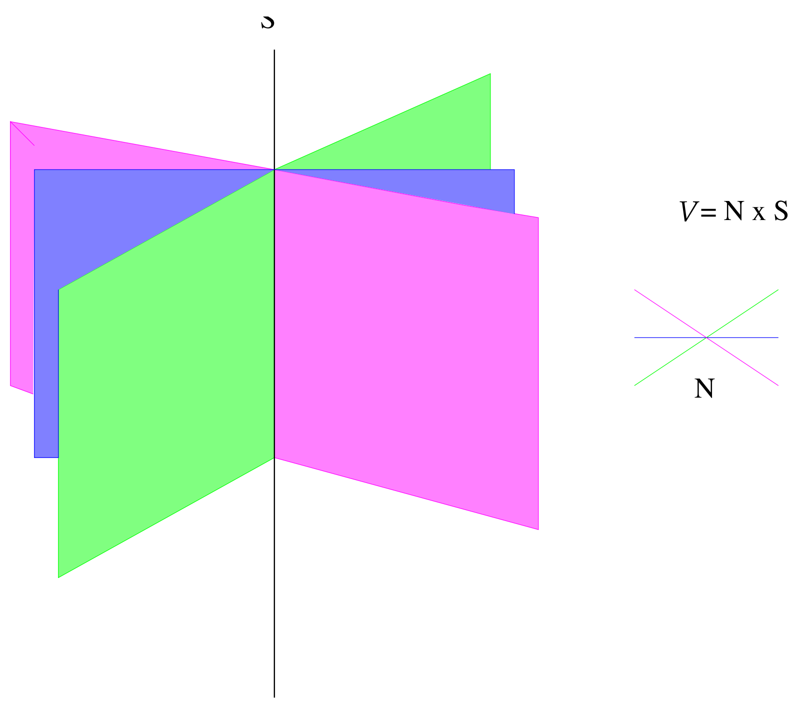

FIGURE 3. An example of three colinear planes and their normal slice.

Let $\tilde{N} = U \cap P$, where U is a neighbourhood of x in M. Then for the cases the book cares about

$$(M^{c+\epsilon}, M^{c-\epsilon}) \simeq (N-\text{data}) \times (T-\text{data}) = (\tilde{N}, \tilde{N} \cap M^{c-\epsilon}) \times (B_k, \partial B_k),$$

where c is the critical level for the critical point z.

The quasi-local cycles are the cycles viewed on the T-data part – $(B_k, \partial B_k)$ – so

$$\begin{aligned}(2\pi i)^d a_r &\sim \int_{C_*} \omega^{-r-1} F(\omega) d\omega \\ &\sim \sum_{z \in \text{contrib}} \int_{C_*(z)} \omega^{-r-1} F(\omega) d\omega \\ &\sim \sum_{z \in \text{contrib}} \int_{C(z)} \left(\int_{N-\text{data}} \omega^{-r-1} F(\omega) d\omega \right).\end{aligned}$$